Single-Seed Traversal of the PMC Graph: A Seven-Vertex Counterexample

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Abstract

Bergé, Limouzy and Schivre ask in §1.4 of [1] whether, with their directed neighbourhood N (Definition 20) kept fixed, one arbitrary potential maximal clique (PMC) suffices as the seed of a BFS that lists all PMCs. We answer this question negatively. Let $G = \overline{K_2} \vee P_5$ be the graph obtained by joining a five-vertex path to two non-adjacent vertices. We prove that G has no clique separator, exactly two minimal triangulations, and exactly six PMCs; that under N every admissible guess satisfies $X^* = V(G)$; and that the resulting twelve arcs leave two PMCs, B and C, with in-degree zero. Hence every single-seed traversal lists at most five of the six PMCs, counting the seed itself. All steps are hand-checkable: the neighbourhood is evaluated under the reading of §2.4, the authors' explicit $X^* = V(G)$ case, and the only outside ingredient is the standard simplicial-vertex property of chordal graphs [3]; we do not assume that the general endpoint question is settled. The example bears on the single-seed perspective of §1.4, not on the multi-seed algorithm of [1]; it does not claim priority, and it does not claim that seven vertices is minimal.

Keywords: potential maximal clique; minimal triangulation; enumeration; graph traversal; counterexample.

1 Introduction

1.1 The question

Section 1.4 of [1] proposes to improve the enumeration of PMCs by traversing a directed graph whose vertices are the PMCs of a graph G and whose arcs are given by a neighbourhood relation N (Definition 20 of [1]). The proposal is to compute one arbitrary PMC of G, put it alone in the starting queue of the traversal, and explore the PMC graph with N; the authors ask whether all PMCs are then enumerated. They report that the traversal did not miss any PMC on the graph classes they tested, and that brute-force checking limited them to at most 20 vertices. The arXiv version of §1.4 calls the traversal a BFS, while the HAL version of the same section calls it a DFS; the wording does not change which PMCs are reachable from a given seed, and we state the question in terms of reachability.

The question is whether, for every graph G and every PMC Ω' of G, the set of PMCs reachable from Ω' along the arcs of N is the whole set of PMCs of G.

We answer this question negatively, with an explicit graph on seven vertices. The example is small enough that every step of the argument can be checked by hand. The neighbourhood is evaluated under the §2.4 reading, that is, the authors' explicit $X^* = V(G)$ case, not a general solution of the endpoint question; apart from that, the only outside ingredient is the standard simplicial-vertex property of chordal graphs [3].

1.2 The result

Theorem 1. Let $G = \overline{K_2} \vee P_5$, with P_5 the path 2-3-4-5-6 and with the two non-adjacent vertices $0, 1$ each adjacent to every vertex of the path. Let N be the neighbourhood of Definition 20 of [1], evaluated under the convention of §2.4. Then G has exactly six PMCs, N has exactly twelve arcs, and two of the PMCs have in-degree zero. Consequently, every single-seed traversal of the PMC graph lists at most five of the six PMCs, even when the seed itself is counted as listed.

The six PMCs are $A = \{0,1,2,3\}$, $B = \{0,1,3,4\}$, $C = \{0,1,4,5\}$, $D = \{0,1,5,6\}$, $E = \{0,2,3,4,5,6\}$ and $F = \{1,2,3,4,5,6\}$. The two PMCs of in-degree zero are B and C. The arcs are $A, B, C, D \to E, F$ and $E, F \to A, D$. Starting from A, B, C, D, E, F one reaches respectively $4, 5, 5, 4, 4, 4$ PMCs, the seed included.

1.3 What is and is not claimed

The example does not refute any theorem of [1]. The multi-seed algorithm of [1] is correct for its own reasons, and nothing here affects it; what fails is the single-seed variant proposed in §1.4. The example depends on the directedness of N: forgetting the orientation, the PMC graph of G is connected, and a single undirected seed would reach all six PMCs. We do not claim priority: we have not found the example in the public literature we checked, but our search was not exhaustive. We do not claim that seven vertices is minimal; we did not search for smaller graphs.

2 Preliminaries

We follow [1] for terminology. All graphs are finite, simple and undirected; $V(G)$ and $E(G)$ denote the vertex and edge sets. For a vertex v, $N_G(v)$ is the set of its neighbours. For a set $S \subseteq V(G)$, the **external neighbourhood** is $N_G(S) = \{v \notin S : \exists s \in S,\ vs \in E(G)\}$; this is the reading under which the formula $S_G(A,b) = N_G(C_G(b, N_G(A)))$ of [1] satisfies $S_G(A,b) \subseteq N_G(A)$, as stated there. We write $G[S]$ for the subgraph induced by S, and $C_G(v, S)$ for the connected component of v in $G - S$.

2.1 Separators

Definition 1 (separator; [1, Definition 8]). Let $a, b \in V(G)$ and $S \subseteq V(G) \setminus \{a, b\}$. Then S is an (a,b)-separator if a and b lie in different connected components of $G - S$; S is a minimal (a,b)-separator if no proper subset of S is an (a,b)-separator.

For a vertex $b \notin S$ we write $C_G[b, S] = S \cup C_G(b, S)$, the union of S with the component of b in $G - S$. For $A \subseteq V(G)$ connected with $a \in A$ and $b \notin N_G[A]$, the **close separator** $S_G(A,b)$ is the unique minimal (a,b)-separator contained in $N_G(A)$ [1, Definition 10 and Lemma 4; 2]. It satisfies $S_G(A,b) = N_G(C_G(b, N_G(A)))$.

Definition 2 (X-close separator; [1, Definition 12]). Let $a, b \in V(G)$ and $X \subseteq V(G)$ with $a \in X$. If $X \setminus \{a\}$ is an (a,b)-separator, the X-close (a,b)-separator is $S_G(A,b)$ with $A = C_G(a, X \setminus \{a\})$.

2.2 Chordal graphs, triangulations and PMCs

A graph is **chordal** if it has no induced cycle of length at least four. A **triangulation** of G is a chordal graph H obtained from G by adding edges; the added edges form the set $F = H \setminus G$. A triangulation is **minimal** if $H - e$ is not chordal for every $e \in F$ ([1, Definition 13]).

Definition 3 (PMC; [1, Definition 14]). A set $\Omega \subseteq V(G)$ is a potential maximal clique of G if it is an inclusion-wise maximal clique of some minimal triangulation of G.

A PMC need not be a clique of G itself; this is exactly what makes the notion useful, and it matters below, where four of the six PMCs contain the non-edge 01.

2.3 The neighbourhood N

Definition 4 (N; [1, Definition 20]). Let Ω' and Ω be PMCs of G. Then $(\Omega', \Omega) \in N$ if Ω is obtained from Ω' by the following process:

1. choose $u^* \in \Omega'$ and set $X^* = \Omega' \cup N_G(u^*)$;
2. choose $b^* \in \Omega'$ and compute the X^*-close (b^*, u^*)-separator S^*;
3. set $\Omega = X^* \cap C_G[b^*, S^*]$.

The paper keeps in N only the pairs for which the computed set Ω is indeed a PMC.

Algorithm 1 (PMC-BFS; [1, Algorithm 2]). Given G and a set $\hat{\Pi}$ of PMCs, put $\Pi = \emptyset$ and $L = \hat{\Pi}$. While $L \neq \emptyset$, remove a PMC Ω' from L, and for each $\Omega \in N(\Omega')$ with $\Omega \notin \Pi \cup \hat{\Pi}$, add Ω to Π and to L. Return Π.

The returned set Π is disjoint from the input set $\hat{\Pi}$: the seeds are not counted in Π. When asking whether a traversal "lists all PMCs", the seed must therefore be added back, since it is already known before the traversal starts.

2.4 Reading Definition 20 at the endpoints

Definition 1 requires an (a, b)-separator to be a subset of $V(G) \setminus \{a, b\}$, and Definition 2 asks that $X \setminus \{a\}$ be an (a, b)-separator. Definition 4 places both u^* and b^* in $\Omega' \subseteq X^*$, and it asks for the X^*-close (b^*, u^*)-separator, that is, $a = b^*$ and $b = u^*$ in the notation of Definition 2.

Combining the three literally: if $u^* \neq b^*$, then $u^* \in X^* \setminus \{b^*\}$, so $X^* \setminus \{b^*\}$ contains u^* and is not a (b^*, u^*)-separator; if $u^* = b^*$, no set separates a vertex from itself. Thus, read literally, step 2 of Definition 4 is unsatisfiable for every graph and every choice of (u^*, b^*), and N would be empty. This is incompatible with Figure 9(a) of [1], which exhibits a PMC Ω that is a neighbour of Ω', and with the use of N in §4.3 of [1].

The paper itself indicates the reading it intends: immediately after Definition 12 it remarks that the close (a, b)-separator, taken without mention of X, "is also the X^*-close (a, b)-separator, with $X^* = V(G)$". We therefore use the following convention, which is the one licensed by that remark: for $X^* = V(G)$, the X^*-close (b^*, u^*)-separator is the ordinary close separator $S_G(\{b^*\}, u^*)$ of [1, Definition 10], defined whenever b^* and u^* are non-adjacent. Every admissible guess in our graph has $X^* = V(G)$ (Lemma 6), so only the ordinary close separator is used in the computation; no general rule of [1] is modified, and no convention is invented for the degenerate cases.

3 The graph

3.1 The graph and its basic properties

Let $V(G) = \{0, 1, 2, 3, 4, 5, 6\}$. The edges are the four path edges 2-3, 3-4, 4-5, 5-6, together with the ten edges 0-i and 1-i for $i = 2, \ldots, 6$. Thus $G = \overline{K_2} \vee P_5$, where $\overline{K_2}$ is the graph on $\{0, 1\}$ with no edge and P_5 is the path 2-3-4-5-6: both 0 and 1 are adjacent to every vertex of the path. The graph has 14 edges and 7 non-edges, namely $01, 24, 25, 26, 35, 36, 46$. The vertices 0 and 1 are false twins: $N_G(0) = N_G(1) = \{2, 3, 4, 5, 6\}$.

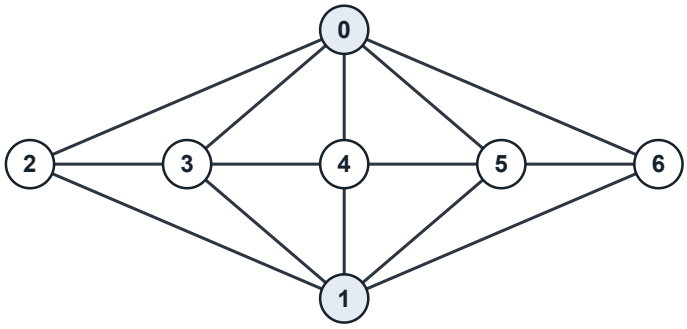

Figure 1. The graph $G = \overline{K_2} \vee P_5$. The path is 2-3-4-5-6; the vertices 0 and 1 are non-adjacent, and each is adjacent to every vertex of the path.

A **clique separator** is a clique T such that $G - T$ is disconnected.

Proposition 2. G has no clique separator.

Proof. Every clique of G contains at most one of $0, 1$ (they are non-adjacent) and at most two vertices of the path, and if it contains two path vertices they are consecutive (the path has no triangle). Let T be a clique and let k be the number of path vertices in T. If T contains no vertex of $\{0, 1\}$, then $k \le 2$; if it contains one, then $k \le 2$ as well. In either case at least three path vertices remain after deleting T. Moreover, T contains at most one of $0, 1$, so at least one of them remains. A remaining vertex of $\{0, 1\}$ is adjacent to every remaining path vertex, so the remaining graph is connected. Hence $G - T$ is connected for every clique T. $\square$

3.2 Two minimal triangulations

Proposition 3. Under the per-edge definition of minimality, G has exactly two minimal triangulations, with added edge sets

$$F_1 = \{01\} \qquad \text{and} \qquad F_2 = \{24, 25, 26, 35, 36, 46\}.$$

Proof. (i) Let $x, y \in \{2, \dots, 6\}$ be non-adjacent in the path. Then 0-x-1-y-0 is an induced 4-cycle of G: its four edges are present, while 01 and xy are not edges. Every chordal supergraph of G must add a chord to this cycle, and the only possible chords are 01 and xy. Hence every triangulation of G either contains 01 or contains all six non-edges of the path.

(ii) Both candidates are chordal. For $G + F_1$, use the following observation: if a vertex u is adjacent to all other vertices and $G - u$ is chordal, then G is chordal, since every cycle of length at least four that passes through u has a chord uz for any vertex z of the cycle other than u and its two cycle-neighbours. In $G + F_1$ the vertex 0 is adjacent to all others, and after deleting it the vertex 1 is adjacent to all others; deleting both leaves P_5, which is chordal. Hence $G + F_1$ is chordal. For $G + F_2 = K_7 - 01$, the vertices $2, 3, 4, 5, 6$ are universal in turn: deleting them one by one leaves the graph on $\{0, 1\}$ with no edge, which is chordal; hence $G + F_2$ is chordal.

(iii) Both are per-edge minimal. Removing 01 from $G + F_1$ gives G, which contains the induced 4-cycle 0-2-1-5-0. Removing $xy \in F_2$ from $G + F_2$ leaves the induced 4-cycle 0-x-1-y-0, whose only possible chords 01 and xy are both absent.

(iv) There is no third minimal triangulation. Let H be a triangulation. If $01 \notin H$, then by (i) all six edges of F_2 lie in H; since these are all the non-edges of G other than 01, we get $H = G + F_2$. If $01 \in H$, write $H = K_2 \vee J$ with $J = P_5 + E'$ and $E' \subseteq F_2$. If $E' = \emptyset$, then $H = G + F_1$. If $E' \neq \emptyset$, then J is a chordal supergraph of P_5 different from P_5, so Lemma 4 (proved in Appendix A) provides a fill edge $e \in E'$ such that $J - e$ is chordal; then $H - e = K_2 \vee (J - e)$ is chordal, so H is not per-edge minimal. Hence the only per-edge minimal triangulations are $G + F_1$ and $G + F_2$. $\square$

Lemma 4 (fill-edge lemma for a path). Let $P = p_1 \cdots p_n$ be a path with $n \le 5$, and let J be a chordal graph on $V(P)$ containing all edges of P. If $J \neq P$, then some fill edge e of J has $J - e$ chordal.

The proof is given in Appendix A. Its length is the only part of the argument that is not a direct check; the reader willing to verify the two minimal triangulations by inspecting the $2^7 = 128$ subsets of non-edges may do so instead.

3.3 The six PMCs

Proposition 5. G has exactly six PMCs, namely

$$A = \{0,1,2,3\}, \quad B = \{0,1,3,4\}, \quad C = \{0,1,4,5\}, \quad D = \{0,1,5,6\},$$
$$E = \{0,2,3,4,5,6\} = V \setminus \{1\}, \quad F = \{1,2,3,4,5,6\} = V \setminus \{0\}.$$

Proof. By Proposition 3 and Definition 3, the PMCs are the maximal cliques of $G + F_1$ and $G + F_2$. In $G + F_1 = K_2 \vee P_5$, the maximal cliques are $\{0,1\}$ together with an edge of the path, that is, A, B, C, D. In $G + F_2 = K_7 - 01$, the maximal cliques are $V \setminus \{0\} = F$ and $V \setminus \{1\} = E$. These six sets are pairwise distinct. □

Note that A, B, C, D contain the non-edge 01; by Definition 3 this is allowed, and it is essential to the example.

4 The neighbourhood of the six PMCs

4.1 All admissible guesses have $X^* = V(G)$

Fix a PMC Ω' of G, a vertex $u^* \in \Omega'$ and a second vertex $b^* \in \Omega' \setminus \{u^*\}$. We call the guess **admissible** if u^* and b^* are non-adjacent: if they are adjacent, no set can separate them, and if $u^* = b^*$ no set separates a vertex from itself, so in either case the guess produces no arc.

Lemma 6. For every admissible guess, $X^* = \Omega' \cup N_G(u^*) = V(G)$.

Proof. (a) Suppose $u^* \in \{0,1\}$. By symmetry assume $u^* = 0$. Then $N_G(0) = \{2,\dots,6\}$, so $X^* = \Omega' \cup \{2,\dots,6\}$. If $1 \in \Omega'$, then $X^* = V(G)$. The only PMC not containing 1 is E. For $\Omega' = E$ we have $X^* = E$, and every $b^* \in E \setminus \{0\} \subseteq \{2,\dots,6\}$ is adjacent to 0, contradicting admissibility.

(b) Suppose $u^* \in \{2,\dots,6\}$. Then u^* is adjacent to both 0 and 1, so $\{0,1\} \subseteq X^*$. Suppose for contradiction that $X^* \neq V(G)$, and choose $v \in V(G) \setminus X^*$. Since $\{0,1\} \subseteq X^*$, we have $v \in \{2,\dots,6\}$. If $\Omega' \in \{E, F\}$, then $v \in \Omega' \subseteq X^*$, a contradiction. Hence $\Omega' = \{0,1\} \cup I$ for some edge I of the path. From $v \notin X^* \supseteq \Omega'$ we get $v \notin I$, and from $v \notin N_G(u^*)$ we get that v is not a path-neighbour of u^*; moreover $u^* \in \Omega' \cap \{2,\dots,6\} = I$. Every $b^* \in \Omega' \setminus \{u^*\}$ is therefore one of $0, 1$, or the other endpoint of I; all of these are adjacent to u^*. This contradicts admissibility. □

4.2 The table of admissible guesses

By Lemma 6 every admissible guess has $X^* = V(G)$, so the X^*-close separator is the ordinary close separator $S_G(\{b^*\}, u^*)$, and $\Omega = C_G[b^*, S^*]$. The following table lists all admissible guesses, grouped by b^*; the second column lists the possible u^*, and the third and fourth columns give S^* and Ω. The table lists fourteen ordered pairs (b^*, u^*) in eight rows; the row for $b^* = 4$ is split in two because its two values of u^* give different separators S^* and different sets Ω.

b^*	admissible u^*	S^*	Ω	PMC?
0	1	$\{2,3,4,5,6\}$	E	yes
1	0	$\{2,3,4,5,6\}$	F	yes

b^*	admissible u^*	S^*	Ω	PMC?
2	4, 5, 6	$\{0,1,3\}$	A	yes
3	5, 6	$\{0,1,4\}$	$\{0,1,2,3,4\}$	no
4	2	$\{0,1,3\}$	$\{0,1,3,4,5,6\}$	no
4	6	$\{0,1,5\}$	$\{0,1,2,3,4,5\}$	no
5	2, 3	$\{0,1,4\}$	$\{0,1,4,5,6\}$	no
6	2, 3, 4	$\{0,1,5\}$	D	yes

Each row is a direct computation; we spell out one. Take $b^* = 2$ and $u^* = 4$. Then $N_G(2) = \{0,1,3\}$, and the component of 4 in $G - \{0,1,3\}$ is $\{4,5,6\}$, whose external neighbourhood is $\{0,1,3\}$; hence $S^* = \{0,1,3\}$, and $C_G[2, S^*] = \{0,1,2,3\} = A$. The other rows are obtained in the same way.

The four sets in the "no" rows are not PMCs by Proposition 5, so by Definition 4 they are discarded and contribute no arc.

4.3 Arcs and in-degrees

Proposition 7. The arcs of N are exactly

$$A, B, C, D \to E, F \qquad \text{and} \qquad E, F \to A, D,$$

twelve arcs in total. The in-degrees are $2, 2, 4, 4, 0, 0$ for A, D, E, F, B, C respectively.

Proof. By Lemma 6 every admissible guess has $X^* = V(G)$, so the guess (u^*, b^*) is available from every PMC that contains both vertices; the table lists the resulting Ω for each pair. Three things follow. First, the pairs with $b^* \in \{0,1\}$ are $\{0,1\}$ and its reverse, and they occur exactly in the PMCs containing both 0 and 1, namely A, B, C, D; these rows produce E and F. Second, the pairs with $b^* \in \{2, \dots, 6\}$ listed in the table occur only in E or F, because E and F contain every path vertex; these rows produce A (from $b^* = 2$) and D (from $b^* = 6$), while the remaining rows produce non-PMC sets, which are discarded. Third, B and C occur in no row. Hence the arcs are $A, B, C, D \to E, F$ and $E, F \to A, D$, twelve in total, with in-degrees $2, 2, 4, 4, 0, 0$ for A, D, E, F, B, C. □

5 Single-seed traversal

5.1 Proof of Theorem 1

Proof. By Proposition 7 the PMCs B and C have in-degree zero: they can occur only as seeds. If the seed is B, the reachable PMCs are B, E, F, A, D, five in total, and C is never reached. If the seed is C, the situation is symmetric and B is never reached. If the seed is A or D, the reachable set is $\{A, D, E, F\}$, and both B and C are missed. If the seed is E or F, the reachable set is again $\{A, D, E, F\}$, and both B and C are missed. In every case at most five PMCs are listed. □

The six coverage numbers, the seed counted, are $4, 5, 5, 4, 4, 4$ for the seeds A, B, C, D, E, F.

5.2 Counting the seed

Algorithm 1 returns a set disjoint from its input: a PMC that has been seen is never added again, so the seed is not returned even when a cycle leads back to it (in our graph, $A \to E \to A$). With a single seed Ω', the returned set is the set of PMCs reachable from Ω' by a path of length at least one, with the seed removed; the total number of PMCs known after the run is the returned set together with the seed. The theorem above counts the seed, so it is the strongest form of the statement: even with the seed counted, one PMC is always missing.

6 Discussion

6.1 Directedness

The orientation of N is essential. If the arcs of N are forgotten, the PMC graph of G is connected (for instance B-E-A-F-C is a path in the underlying undirected graph), so a single undirected seed reaches all six PMCs. The example therefore illustrates a limitation of the directed relation N of [1], not of graph search in general.

6.2 Relation to the literature

The single-seed proposal appears only in §1.4 of [1], as a perspective; the paper's algorithm starts from the PMCs produced by the first three steps of the BT-Enum scheme, and its correctness does not rest on single-seed reachability. The example is therefore a counterexample to a proposed simplification, not to a theorem.

The example also shows that the PMC graph of a graph with no clique separator can have several sources. Whether every PMC is reachable from the set of sources is a different question, which we do not address.

6.3 Limitations

We do not claim that seven vertices is minimal. We do not claim priority: the literature we checked (the arXiv and HAL versions of [1], the JGA 2025 slides, the M2 report of one author, and the papers cited there) contains no equivalent example, but our search was not exhaustive. The example does not settle the polynomial-delay question of [1]; it only removes the single-seed shortcut from the list of candidate approaches.

6.4 Verification materials

Two scripts accompany the preparation of this report as cross-checks, not as proofs. The first enumerates all $2^7 = 128$ subsets of non-edges, keeps the chordal ones, extracts the inclusion-minimal triangulations, and computes the neighbourhood by the three steps of Definition 4 under the $X = V(G)$ convention used here. The second is an independent check that compares two enumeration routes (one through minimal triangulations, one through components and minimal separators); for each candidate it enumerates the inclusion-minimal separators and takes the component on the b^* side that is minimal by inclusion, then verifies the resulting PMC list and arcs. Both report the same six PMCs, twelve arcs, the two sources B and C, and the coverage numbers $4, 5, 5, 4, 4, 4$. Neither script implements the general Definition 20 for arbitrary graphs, and neither is part of the proof.

References

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Appendix A. The fill-edge lemma for a path

Lemma 4. Let $P = p_1 \cdots p_n$ be a path with $n \leq 5$, and let J be a chordal graph on $V(P)$ containing all edges of P. If $J \neq P$, then some fill edge e of J has $J - e$ chordal.

Criterion. Let J be chordal and let $xy \in E(J)$. Then $J - xy$ is not chordal if and only if x and y have two non-adjacent common neighbours.

Proof of the criterion. If z, w are common neighbours of x, y with $zw \notin E(J)$, then x-z-y-w-x is an induced 4-cycle of $J - xy$. Conversely, let C be an induced cycle of length at least four in $J - xy$. Since J is chordal, C has a chord in J; every chord of C other than xy would also be present in $J - xy$, so the only chord is xy. Hence $C = x$-P_1-y-P_2-x for the two arcs of C. If one arc had length at least three, then that arc together with the edge xy would form an induced cycle of J of length at least four, contradicting chordality. Hence both arcs have length two, and their internal vertices are non-adjacent common neighbours of x and y. □

Proof of Lemma 4. We induct on n. For $n \le 2$ there are no fill edges. For $n = 3$, the only possible fill edge is p_1p_3, and deleting it gives P. Let $n \ge 4$ and let v be a simplicial vertex of J, which exists because J is chordal [3].

If v is incident with a fill edge vx, then the common neighbours of v and x lie in $N_J(v)$, which is a clique because v is simplicial; by the criterion $J - vx$ is chordal.

Otherwise v is incident only with path edges. If v is an endpoint of P, then $J - v$ is a chordal supergraph of the shorter path $P - v$. If $J - v$ were equal to that path, then $J = P$, contrary to the hypothesis; so by induction $J - v$ has a fill edge e with $(J - v) - e$ chordal. Since v is a pendant vertex of J, adding it back preserves chordality, so $J - e$ is chordal.

If v is an internal vertex, with path neighbours x, y, then $xy \in E(J)$ because v is simplicial, and xy is a fill edge. Delete v and regard xy as an edge of the shortened path P'. If $J - v$ has no fill edge beyond those of P', then $J = P + xy$ and deleting xy gives P. Otherwise, by induction applied to the chordal supergraph $J - v$ of P', there is a fill edge $e \neq xy$ of $J - v$ such that $(J - v) - e$ is chordal; the neighbourhood of v in J is the clique $\{x, y\}$, so adding v back preserves chordality and $J - e$ is chordal. □